Cotangent complex

Given a map of ordinary commutative rings $A \to B$ one has a B-module of <u>relative Kähler differentials</u> given by:

$$\Omega_{B/A} := \langle db \mid b \in B \rangle \Big/ \left\{ \begin{array}{l} d(b+b') = db + db' \\ d(bb') = b\,db' + b'\,db \; (\star) \\ da = 0 \quad \forall a \in A \end{array} \right\}.$$

This B-module has the following universal property: for any M a B-module & $d_M : B \to M$ an <u>A-linear derivation of B into M</u>, i.e. d_M is A-linear and $d_M(bb') = d_M(b) \cdot b' + b \cdot d_M(b')$ $(\star)$ one has an unique factorization:

$$\begin{array}{ccc} B & \xrightarrow{d} & \Omega_{B/A} \\ & \searrow^{d_M} & \downarrow{\varphi_M} \\ & & M \end{array}$$

where φ_M is a B-module map.

In other words, $\quad \mathrm{Der}_A(B, M) \simeq \mathrm{Hom}_{\mathrm{Mod}_B}(\Omega_{B/A}, M)$.

To ~~[illegible]~~ define a similar object in derived geometry we are faced w/ the problem that we can't write the Leibniz equations $(\star)$.

The following observations come to the rescue:

Lemma 1: For any B-module M one has an equivalence:

$$\mathrm{Der}_A(B, M) \simeq \mathrm{Hom}_{\mathrm{Mod}_B}(\Omega_{B/A}, M) \simeq \mathrm{Hom}_{\mathrm{CAlg}_{A/-}}(B, B \oplus M),$$

where $B \oplus M \to B$ is a map of A-algebras where $B \oplus M$ is an algebra w/ the square-zero extension structure, i.e.

$$\begin{array}{l} (B \oplus M) \times B \oplus M \to B \oplus M \\ (b, m) \quad (b', m') \mapsto (bb', b'm + bm'). \qquad (\nabla) \end{array}$$

So Lemma 1 changes to problem of writing eq. $(\star)$ to writing (∇), i.e.

endowing $B \oplus M$ w/ a commutative ring structure. Now the second observation comes to the rescue:

Lemma 2: One has an equivalence of categories:

$$\mathrm{Mod}_B \simeq \mathrm{Ab}(\mathrm{CAlg}_{/B}), \quad \text{where Ab means abelian group objects in } \mathrm{CAlg}_{/B}.$$
$$M \mapsto B \oplus M.$$
$$\ker \alpha \leftarrow\!\!\mapsto \alpha : B' \to B.$$

Rk: The passage $B \in \mathrm{CAlg} \rightsquigarrow \mathrm{CAlg}_{/B} \rightsquigarrow \mathrm{Ab}(\mathrm{CAlg}_{/B})$ can be performed in great generality and the result is always an abelian category.

Thus, we now have a result that is more amenable to generalizations for derived rings.

For a second we will consider the category of CAlg^{nc}. non-connective algebras. ~~Let $A \in \mathrm{CAlg}^{nc}$ one has~~

Prop: Let $A \in \mathrm{CAlg}^{nc}$ we have an equivalence of ∞-categories:

$$\mathrm{Spctr}(\mathrm{CAlg}^{nc}_{/A}) \simeq \mathrm{Mod}_A. \quad , \text{ where } \mathrm{Spctr}(\mathrm{CAlg}^{nc}_{/A}) \text{ means spectrum objects in } \mathrm{CAlg}^{nc}_{/A}.$$

In particular, one has:

$$\mathrm{Mod}_A^{\leq 0} \simeq \mathrm{Com.Monoid}(\mathrm{CAlg}_{A//A}).$$

We will denote by:
$$\overset{\text{Split sqz.}}{} \quad \mathrm{QCoh}(S)^{\leq 0} \to \mathrm{Sch}^{aff}_{S/-}$$
$$\mathcal{F} \mapsto \mathrm{Spec}(A \oplus \Gamma(S, \mathcal{F})) =: S_{\mathcal{F}}$$
the affine scheme corresponding to $A \oplus \Gamma(S, \mathcal{F})$, where $S = \mathrm{Spec}\, A$.
[See [HA. Thm 7.3.4.13] or [GR-II, Chapter 6 Prop. 1.8.3].

Rk: The following is a consequence of the equivalence in the Proposition.
Let $M \in \mathrm{Mod}_A$ and consider $A \oplus M$ the image via

$$\mathrm{Mod}_A = \mathrm{Spctr}(\mathrm{CAlg}^{nc}_{/A}) \to \mathrm{CAlg}^{nc}_{/A}. \quad \text{The multiplication:}$$

$$m: \cancel{(M \oplus A)} (A \oplus M) \otimes (A \otimes M) \to A \oplus M$$ ~~[illegible]~~

(i) when restricted to $A \otimes A$ is homotopic to the multiplication at A.
(ii) ——————— $A \otimes M$ or $M \otimes A$ is homotopic to the A-module str. on M
(iii) ——————— $M \otimes M$ is null homotopic, i.e. homotopic to zero.

In particular, for $A \in \mathrm{CAlg}$, $M \in \mathrm{Mod}_A^{\leq 0}$ one has.

$$H^*(A \oplus M) \simeq H^*(A) \oplus H^*(M)$$ w/ algebra structure given by the split square-zero extension of $H^*(A)$ by $H^*(M)$.

Instead of defining the analogue of $\mathcal{R}_{B/A}$ for derived rings we will consider the general situation of prestacks. We need a couple of preliminaries.

Given $\mathcal{X} \in \mathrm{PStk}$ we let $\mathrm{QCoh}(\mathcal{X}) := \lim\limits_{\substack{S \to \mathcal{X} \\ S \in \mathrm{Sch}^{\mathrm{aff}}}} \mathrm{QCoh}(S)$.

here the limit is taken in the category of stable ∞-categories (or DG-categories) w/ continuous functors with respect to the pullback functor.

For $\mathcal{X} \in \mathrm{PStk}$ and $x : S \to \mathcal{X}$ a point, we let

$$\mathrm{Lift} : \mathrm{QCoh}(S)^{\leq 0} \to \mathrm{Spc}.$$
$$\mathcal{F} \mapsto \mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}) = \left\{ \begin{array}{ccc} S_{\mathcal{F}} & & \\ \uparrow & \overset{\bar{x}}{\searrow} & \\ S & \xrightarrow{x} & \mathcal{X} \end{array} \right\}$$ i.e. the space of lifts of x to $\bar{x}$.

Moreover, given $\mathcal{F}_1 \to \mathcal{F}_2$ a map in $\mathrm{QCoh}(S)^{\leq 0}$ s.t. $H^0(\mathcal{F}_1) \twoheadrightarrow H^0(\mathcal{F}_2)$ is surjective we have.

$$\mathcal{F} := 0 \underset{\mathcal{F}_2}{\times} \mathcal{F}_1 \in \mathrm{QCoh}(S)^{\leq 0}.$$

Notice that $\mathcal{F} \mapsto S_{\mathcal{F}}$ sends pullbacks to pushouts, so we have:

$$S_{\mathcal{F}} \simeq S_0 \underset{S_{\mathcal{F}_2}}{\amalg} S_{\mathcal{F}_1} = S \underset{S_{\mathcal{F}_2}}{\amalg} S_{\mathcal{F}_1}.$$

We finally have:

Def'n: A prestack $\mathcal{X}$ admits a cotangent space at $x : S \to \mathcal{X}$ if

$$\mathrm{Maps}_{S/-}(S_{\mathcal{F}}, \mathcal{X}) \xrightarrow{\simeq} \cancel{\mathrm{Maps}\ \mathcal{X}} \underset{\mathrm{Maps}_{S/}(S_{\mathcal{F}_1}, \mathcal{X})}{\times} \mathrm{Maps}_{S/-}(S_{\mathcal{F}_1}, \mathcal{X}). \quad (\square)$$

The above implies that we can extend Lift to a functor:

$$\widetilde{\mathrm{Lift}} : QCoh(S)^{-} := \bigcup_{n \geq 0} QCoh(S)^{\leq n} \to Spc \quad \text{by}$$

$$\mathcal{F} \in QCoh(S)^{\leq n} \mapsto \Omega^i(\mathrm{Maps}_{S/-}(S_{\mathcal{F}[i]}, \mathcal{X})) \quad \text{for any } i \geq n.$$

Notice the isomorphism $(\square)$ guarantees that $\widetilde{\mathrm{Lift}}$ is well-defined.

We let $T^*_x \mathcal{X} \in QCoh(S)^{\circledast}$ [Technically speaking this might be a pro-object in QCoh(S).] s.t.

$$\mathrm{Hom}_{QCoh(S)^{-}}(T^*_x \mathcal{X}, \mathcal{F}) \simeq \Omega^i(\mathrm{Maps}_{S/-}(S_{\mathcal{F}[i]}, \mathcal{X})) \quad \cancel{\text{where}}$$

for some $i \in \mathbb{N}$ s.t. $\mathcal{F}[i] \in QCoh(S)^{\leq 0}$.

Notice for any pullback
$$\begin{array}{ccc} \mathcal{F}_1' & \to & \mathcal{F}_1 \\ \downarrow & & \downarrow \\ \mathcal{F}_2' & \to & \mathcal{F}_2 \end{array}$$
diagram in $QCoh(S)^{\circledast}$, where all objects are in $QCoh(S)^{\leq 0}$.

one has:

$$\mathrm{Maps}_{S/-}(S_{\mathcal{F}_1'}, \mathcal{X}) \xrightarrow{\simeq} \mathrm{Maps}_{S/}(S_{\mathcal{F}_2'}, \mathcal{X}) \underset{\mathrm{Maps}_{S/}(S_{\mathcal{F}_2}, \mathcal{X})}{\times} \mathrm{Maps}_{S/}(S_{\mathcal{F}_1}, \mathcal{X}).$$

Before giving some examples we introduce two pieces of theory. First we observe that the construction of split square-zero extensions is functorial, i.e. given $f : S_1 \to S_2$ a map of affine schemes one has a commutative diagram:

$$\begin{array}{ccc} QCoh(S_1)^{\leq 0} & \xrightarrow{f_*} & QCoh(S_2)^{\leq 0} \\ \downarrow{\scriptstyle \mathrm{Split}\,\mathrm{Sq}\,Z.} & & \downarrow{\scriptstyle \mathrm{Split}\,\mathrm{Sq}\,Z.} \\ Sch^{aff}_{S_1/-} & \xrightarrow{(-)\coprod_{S_1} S_2} & Sch^{aff}_{S_2/-} \end{array}$$

i.e. $(S_2)_{f_*(\mathcal{F})} = (S_1)_{\mathcal{F}} \coprod_{S_1} S_2$ $\forall\, \mathcal{F} \in QCoh(S_1)^{\leq 0}$.

Thus, ~~the~~ ~~map~~ for $x_2: S_2 \to \mathcal{X}$ & $x_1 := x_2 \circ f$ one has a map:

$$\mathrm{Maps}_{S_2/-}((S_2)_{f_*(\mathcal{F})}, \mathcal{X}) \to \mathrm{Maps}_{S_1/-}((S_1)_{\mathcal{F}}, \mathcal{X}) \quad (\Delta'), \text{ i.e.}$$

"is" "is"

$$\mathrm{Hom}_{QCoh(S_2)^-}(T^*_{x_2}\mathcal{X}, f_*(\mathcal{F})) \to \mathrm{Hom}_{QCoh(S_1)^-}(T^*_{x_1}\mathcal{X}, \mathcal{F}).$$

by the (f^*, f_*) adjunction

$$\simeq \mathrm{Hom}_{QCoh(S_2)^-}(f^* T^*_{x_2}\mathcal{X}, \mathcal{F})$$

So

$$T^*_{x_1}\mathcal{X} \to f^* T^*_{x_2}\mathcal{X}. \quad (\Delta)$$

Def'n: We say $\mathcal{X}$ admits a cotangent complex if for all $x_2: S_2 \to \mathcal{X}$ and any $f: S_1 \to S_2$ as above (Δ) is an isomorphism, equivalently (Δ') is an isomorphism.

~~As usual one says $\mathcal{X}$ has a cotangent complex / space if it has~~

By definition if $\mathcal{X}$ has a cotangent complex it is an object of $QCoh(\mathcal{X})$.

The relative situation is also useful.

Def'n: Let $f: \mathcal{X} \to \mathcal{Y}$ be a map of prestacks we say f admits a cotangent space (resp. complex) at $x: S \to \mathcal{X}$ if one has an object $T^*(\mathcal{X}/\mathcal{Y})$ s.t.

$$\mathrm{Hom}_{QCoh(S)^-}(T^*_x(\mathcal{X}/\mathcal{Y}), \mathcal{F}) \simeq \mathrm{Maps}_{S/-}(S_{\mathcal{F}}, \mathcal{X}) \times_{\mathrm{Maps}_{S/-}(S_{\mathcal{F}}, \mathcal{Y})} *.$$

When $\mathcal{X}$ & $\mathcal{Y}$ admit cotangent spaces we have:

$$T^*_x(\mathcal{X}/\mathcal{Y}) \simeq \mathrm{Cofib}(T^*_y\mathcal{Y} \to T^*_x\mathcal{X}) \quad \text{where } y := f\circ x.$$

Exercise: $f: \mathcal{X} \to \mathcal{Y}$ has a cotangent space iff $\forall\, S \to \mathcal{Y}$ $\mathcal{X} \times_{\mathcal{Y}} S$ has a cotangent space.

Examples: (i) Consider $f: \operatorname{Spec} B \to \operatorname{Spec} A$ we want to compute $T^* \operatorname{Spec} B/\operatorname{Spec} A$ when $B \simeq \operatorname{Sym}_A(V)$ for $V \in \operatorname{Mod}_A$.

let $x: \operatorname{Spec} R \to \operatorname{Spec} B$ & $\tilde{x}: \operatorname{Spec} R\oplus M \to \operatorname{Spec} A$ corresponding to the commutative diagram:

$$\begin{tikzcd} B & A \arrow[l] \arrow[d, "\tilde{x}"] \\ R \arrow[u, leftarrow, "x"] & R\oplus M \arrow[l] \arrow[ul, leftarrow, dashed] \end{tikzcd}$$

We know:

$$\operatorname{Hom}_{\operatorname{Mod}_R}(T^* \operatorname{Spec} B/\operatorname{Spec} A, M)$$

is

$$\operatorname{Maps}_{S/-}(S_M, \operatorname{Spec} B) \times_{\operatorname{Maps}_{S/-}(S_M, \operatorname{Spec} A)} * \qquad (\star)$$

We compute: $\operatorname{Maps}_{S/-}(S_M, \operatorname{Spec} B) \simeq \operatorname{Hom}_{\operatorname{CAlg}_{/R}}(B, R\oplus M)$

$$\operatorname{Maps}_{S/-}(S_M, \operatorname{Spec} A) = \operatorname{Hom}_{\operatorname{CAlg}_{/R}}(A, R\oplus M).$$

So from the diagram we see that $(\star)$ is equivalent to:

$$\operatorname{Fib}\left(\operatorname{Hom}_{\operatorname{CAlg}_{A/-}}(B, R\oplus M)_{\tilde{x}} \to \operatorname{Hom}_{\operatorname{CAlg}_{A/-}}(B, R)\right).$$

Since $B \simeq \operatorname{Sym}_A(V)$ one has

$$\simeq \operatorname{Fib}\left(\operatorname{Hom}_{\operatorname{Mod}A}(V, R\oplus M) \to \operatorname{Hom}_{\operatorname{Mod}A}(V, R)\right)$$

$$\simeq \operatorname{Hom}_{\operatorname{Mod}A}(V, M) \simeq \operatorname{Hom}_{\operatorname{Mod}B}(V \otimes_A B, M).$$

Thus, $\mathbb{L}_{B/A} := T^* \operatorname{Spec} B/\operatorname{Spec} A \simeq V \otimes_A B$.

As with the usual module of Kähler differentials given.

Prop: $A \to B \to C$ morphisms in CAlg. One obtains:

- $\mathbb{L}_{B/A} \otimes_B C \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B}$. a fiber/cofiber sequence in Mod_C

Also for

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & B' \end{tikzcd}$$

a pullback square one has:

$$\mathbb{L}_{B/A} \otimes_B B' \simeq \mathbb{L}_{B'/A'}.$$

<u>Example (ii):</u> Consider $A \to A \oplus A[n]$ for $n \leq 0$.

We want to compute $\mathbb{L}_{A \oplus A[n]/A}$.

<u>Case (i):</u> n is odd, then $A \oplus A[n] \simeq \mathrm{Sym}_A(A[n])$.

$$\text{So } \mathbb{L}_{A \oplus A[n]/A} \simeq (A \oplus A[n]) \otimes_A A[n]. \simeq A[n] \oplus A[2n]$$

<u>Case (ii):</u> n is even. Notice: for $n \geq 1$
$(n \geq 2)$.

$$\begin{tikzcd} A \oplus A[n-1] \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & A \oplus A[n] \end{tikzcd} \quad (\Delta)$$

is a pullback in CAlg.

$$\text{So } \mathbb{L}_{A \oplus A[n]/A} \simeq \mathbb{L}_{A/A \oplus A[n-1]} \otimes_A (A \oplus A[n])$$

So it is enough to calculate. $\mathbb{L}_{A/A \oplus A[n-1]}$.

Consider. the maps: $A \to A \oplus A[n-1] \to A$, then

$$A \otimes_{A \oplus A[n-1]} \mathbb{L}_{A \oplus A[n-1]/A} \to \mathbb{L}_{A/A} \to \mathbb{L}_{A/A \oplus A[n-1]}.$$

$$\text{is} \qquad \text{is}$$

$$A \otimes_{A \oplus A[n-1]} A[n-1] \oplus A[2n-2]. \to 0 \to \mathbb{L}_{A/A \oplus A[n-1]}.$$

$$\text{is} \quad A[n-1]$$

$$\Rightarrow \quad \mathbb{L}_{A/A \oplus A[n-1]} = A[n \text{[illegible]}] \text{[illegible]}.$$

$$\text{So, } \quad \mathbb{L}_{A[n]/A} \simeq (A[n \text{[illegible]}] \text{[illegible]}) \otimes_A (A \oplus A[n]).$$

$$\mathbb{L}_{A[n]/A} \simeq A[n \text{[illegible]}] \oplus A[2n \text{[illegible]}].$$

<u>Case (iii):</u> $\mathbb{L}_{A \oplus A/A} \simeq A.$ using (Δ) w/ $n = 1$.

We will list a result called connectivity estimate. It is a rather useful result to compute & undertand $\mathbb{L}_{B/A}$.

We say a B-module M is <u>n-connective</u> if $H^i(M)=0 \ \forall i > -n$.

For instance, 0-connective $\Leftrightarrow$ $M \in \mathrm{Mod}_B^{\leq 0}$.

Given a morphism $f: M \to N$ in Mod_B f is <u>n-connective</u> if $\mathrm{Fib}(f)$ is n-connective.

Notice f n-connective $\Rightarrow$ $H^{-n}(M) \twoheadrightarrow H^{-n}(N)$ &
$H^i(M) \xrightarrow{\simeq} H^i(N)$ for $i \geq -n+1$.

<u>Thm</u> (HA.7.4.3.1): For $f: A \to B$ in CAlg.

$\mathrm{Gfib}(f)$ n-connective $\Rightarrow$ $\exists$ a $(2n)$-connective map
$\alpha_f: B \otimes_A \mathrm{Gfib}(f) \to \mathbb{L}_{B/A}$.

Let's consider some consequences of this theorem.

<u>Cor 1</u>: Let $f: A \to B$ s.t. $\mathrm{Gfib}(f)$ is n-connective, i.e. $\mathrm{Gfib}(f) \in \mathrm{Mod}_B^{\leq -n}$.
Then $\mathbb{L}_{B/A}$ is n-connective.
If $\mathbb{L}_{B/A}$ is n-connective & ${}^0f: H^0(A) \xrightarrow{\simeq} H^0(B)$, then $\mathrm{Gfib}(f)$ is n-connective

<u>Pf.</u> Consider the fiber seq.

$$\mathrm{Fib}(\alpha_f) \to B \otimes_A \mathrm{Gfib}(f) \to \mathbb{L}_{B/A},$$

then $B \otimes_A \mathrm{Gfib}(f) \in \mathrm{Mod}_B^{\leq -n}$, $\mathrm{Fib}(\alpha_f) \in \mathrm{Mod}_B^{\leq -2n} \subseteq \mathrm{Mod}_B^{\leq -n+1}$

$\Rightarrow \mathbb{L}_{B/A} \in \mathrm{Mod}_B^{\leq -n}$.

The converse follows by an argument by contradiction.

Thus, we have:

Cor 2: For any $S \in Sch^{aff}$, $T^* S \in QCh(S)^{\le 0}$.

Cor 3: For $f : S \to T$ a map in Sch^{aff}, then f is an equivalence iff:

(i) ${}^{cl}f : {}^{cl}S \xrightarrow{\simeq} {}^{cl}T$;

(ii) $T^*(S/T) \simeq 0$.